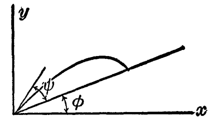
SOLUTION BY A. M. HARDING, University of Arkansas.



The position of the projectile at any time is given by

$$x = v\cos(\varphi + \psi)\cdot t,$$
$$y = v\sin(\varphi + \psi)\cdot t - \tfrac{1}{2}gt^2.$$

Eliminating t we obtain, as the equation of the path,

$$y = x\tan(\varphi + \psi) - \frac{gx^2}{2v^2\cos^2(\varphi + \psi)}. \tag{1}$$

Let r be the range up the plane. Then the coördinates of the point where the projectile meets the inclined plane are

$$x = r\cos\varphi, \qquad y = r\sin\varphi. \tag{2}$$

Substituting these in (1), we find

$$r = \frac{2v^2}{g}\cdot\frac{\cos(\varphi + \psi)\sin\psi}{\cos^2\varphi}. \tag{3}$$

From (1) we find

$$\frac{dy}{dx} = \tan(\varphi + \psi) - \frac{gt}{v\cos(\varphi + \psi)}. \tag{4}$$

This gives the direction at any time t. The projectile strikes the plane after a time

$$t = \frac{r\cos\varphi}{v\cos(\varphi + \psi)} = \frac{2v\sin\psi}{g\cos\varphi}.$$

Hence, when the projectile strikes the plane, we have

$$\frac{dy}{dx} = \tan(\varphi + \psi) - \frac{2\sin\psi}{\cos\varphi\cos(\varphi + \psi)}$$
$$= \frac{\sin(\varphi + \psi)\cos\varphi - 2\sin\psi}{\cos\varphi\cos(\varphi + \psi)}.$$

If the projectile strikes perpendicularly, $dy/dx = -\cot\varphi$. Hence,

$$\frac{\sin(\varphi + \psi)\cos\varphi - 2\sin\psi}{\cos\varphi\cos(\varphi + \psi)} = -\frac{\cos\varphi}{\sin\varphi}.$$

From this equation, we find $\tan\psi = \frac{1}{2}\cot\varphi$. Substituting in (3), we find

$$r = \frac{2v^2\sin\varphi}{g(1 + 3\sin^2\varphi)}.$$

Also solved by H. C. FEEMSTER, H. S. UHLER, G. PAASWELL, C. N. SCHMALL, O. S. ADAMS, HORACE OLSON, and L. A. WARREN.

NUMBER THEORY.

242. Proposed by NORMAN ANNING, Chilliwack, B. C.

Find a function of n which is equal to A_k when $n \equiv k \pmod p$, $k = 1, 2, 3, \cdots, p$.

SOLUTION BY THE PROPOSER.

Let θ be a primitive root of $x^p - 1 = 0$, then it is known that $\theta, \theta^2, \theta^3, \cdots, \theta^p$ are all the pth roots of unity and that $\theta^m + \theta^{2m} + \theta^{3m} + \cdots + \theta^{pm} = p$ or 0, according as m is or is not divisible by p. [See, for example, Burnside and Panton, *Theory of Equations*, Vol. I, pp. 95 and 96.] Consider now the expression:

$$\begin{aligned} f = \frac{1}{p}\,[&A_1\{\theta^{n-1} + \theta^{2(n-1)} + \cdots + \theta^{p(n-1)}\} \\ &+ A_2\{\theta^{n-2} + \theta^{2(n-2)} + \cdots + \theta^{p(n-2)}\} \\ &+ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ &+ A_p\{\theta^{n-p} + \theta^{2(n-p)} + \cdots + \theta^{p(n-p)}\}]. \end{aligned}$$

Since, for any integral values of n, the numbers, $(n - 1)$, $(n - 2)$, $\cdots$, $(n - p)$, are p consecutive integers, one and only one of them is divisible by p.

Suppose $(n - k)$ is divisible by p, *i. e.*, $n \equiv k \pmod{p}$. Then the coefficient of A_k is p and all the other coefficients are zero.

Hence, when $n \equiv k \pmod{p}$, $f = A_k$ as required.

244. Proposed by CLIFFORD N. MILLS, Brookings, South Dakota.

Determine the rational value of x that will render $x^2 + px + q$ a perfect square. What value of x will render $x^2 - 7x + 2$ a perfect square?

Solution by Harold T. Davis, Colorado Springs, Colorado.

Let $x^2 + px + q = y^2$. Then $x^2 + px + (p^2/4) + q - (p^2/4) - y^2 = 0$, or $(2x + p)^2 - (2y)^2 = p^2 - 4q$.

(1) Let $2x + p = z$ and $2y = w$. Then $z^2 - w^2 = p^2 - 4q$; or, if $4q > p^2$, $w^2 - z^2 = 4q - p^2$. Let a and b be complementary factors of $p^2 - 4q$. Then $z + w = a$ and $z - w = b$. Whence $z = (a + b)/2$ and $w = (a - b)/2$. Substituting these values in equations (1), we have

$$x = \frac{a + b - 2p}{4} \text{ and } y = \frac{a - b}{4}.$$

Example. $x^2 - 7x + 2 = y^2$. Here $p^2 - 4q = 41$, the complementary factors of which are 41 and 1. Hence,

$$x = \frac{41 + 1 + 14}{4} = 14 \quad \text{and} \quad y = \frac{41 - 1}{4} = 10.$$

A complete discussion of the solution of the general equation of the second degree in two variables is given in Chrystal's *Algebra*, Part II, page 458.

Also solved by Horace Olson, Norman Anning, H. N. Carleton, O. S. Adams, J. A. Colson, J. L. Riley, N. Pandya and J. H. Weaver.

QUESTIONS AND DISCUSSIONS.

Send all communications to U. G. Mitchell, University of Kansas, Lawrence.

REPLIES.

33. Under what conditions or to what extent is Mr. Iwerson's construction, given below, a useful or practical approximation to a true ellipse? What criterion can be given to measure definitely the degree of approximation?

Mr. Iwerson's approximate construction for an ellipse by ruler and compasses alone, having given the axes, was given in the November, 1916, issue of the Monthly, pp. 354, 355. The following corrections should be made: In the last two lines on p. 354, Ox should be OY, and Oy should be OX.

Note. In the February issue of the Monthly (pp. 90–92), we published a reply to this question by Professor Capron, of the U. S. Naval Academy. Before the February issue had come from the press, Professor Howland, of Wesleyan University, sent in the reply printed below. These two discussions are, accordingly, independent and from entirely different points of view. Professor Capron took as his primary measure of approximation the proportional errors in the radii of curvature at important points. Professor Howland has taken as his measure of approximation the ratio of the distance between the true and constructed curves (measured vertically or normally) to the semi-major axis. The two discussions seem to overlap in but one place. What Professor Capron has called the proportional error in the length of the minor axis and designated as E_1, corresponds to a maximum value of Professor Howland's relative divergence, d_2/a, which for some eccentricities occurs at $x = 0$. There is some slight difference in the formulas since in the one case the error is given relative to the semi-minor and in the other case relative to the semi-major axis.